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# A Uniqueness Theorem for Ordinary Differential Equations

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Abstract. The uniqueness theorem of this paper answers an open question for a system of differential equations arising in a certain n-body problem of classical electrodynamics. The essence of the result can be illustrated using the scalar prototype equation $x' = g_1(x) + g_2(t + x)$ with $x(0) = 0$. The solution of the latter will be unique provided $g_1$ and $g_2$ are continuous positive functions of bounded variation.

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The theorem proved in this paper presents a criterion weaker than a Lipschitz condition which assures uniqueness of solutions of a system of ordinary differential equations. It was designed to resolve an open question in classical electrodynamics described at the end of the paper.

Before stating the theorem let us illustrate it with two scalar examples typifying the problems we had in mind. These examples are easily treated with the theorem which follows. We are unaware of any previous uniqueness theorem which would handle them or the electrodynamics problem of Example 3.

Example 1. If $g_1$ and $g_2$ are continuous positive functions of bounded variation on an open interval containing 0, then the equation

$$x' = g_1(x) + g_2(t + x) \quad \text{with} \quad x(0) = 0$$

has a unique solution on some open interval containing 0.

Example 2. The equation

$$x' = (t + x^{5/3})^{1/3} \quad \text{for} \quad t \geq 0 \quad \text{with} \quad x(0) = 0$$

has a unique solution.

The theorem itself treats a system of $n$ ordinary differential equations

(1)
$$x' = f(t, x)$$

with initial conditions

(2)
$$x(t_0) = x_0.$$

Let  S  be a subset (not necessarily open) of  $R^{n+1}$,
and let  f: S → $R^n$.  Then, given  $(t_0, x_0) \in S$, a solution of
Eqs. (1) and (2) is defined as any differentiable function
x  on an interval  J  such that  $(t, x(t)) \in S$  and
$x' = f(t, x(t))$  for  $t \in J$, while  $t_0 \in J$  and  $x(t_0) = x_0$.
(If  J  contains either of its endpoints, x'(t) is a one-sided
derivative there.)

The norm used in this paper for a vector  $\xi \in R^n$  is
$$||\xi|| = \Sigma_{i=1}^{n} |\xi_i|.$$

Theorem.  Let  f: S → $R^n$  be continuous and satisfy the
following condition. Each point in  S  has an open neighbor-
hood  U, a constant  K > 0,  an integer  m ≥ 0,  and functions
$h_j$  and  $g_j$  for  j = 1, ..., m  such that

(3)  $||f(t,\xi) - f(t,\eta)|| \le K||\xi - \eta|| + K \sum_{j=1}^{m} |g_j(h_j(t,\xi)) - g_j(h_j(t,\eta))|$

on  U ∩ S,  where  $h_j$: U → R  is continuously differentiable
with

(4)  $\dfrac{\partial h_j(t,\xi)}{\partial t} + \sum_{i=1}^{n} \dfrac{\partial h_j(t,\xi)}{\partial \xi_i} f_i(t,\xi) \neq 0$   on  U ∩ S

and each  $g_j$: R → R  is continuous and is of bounded variation
on bounded subintervals.  Then Eqs. (1) and (2) with any point
$(t_0, x_0) \in S$  have at most one solution on any interval  J.

Remarks.  The theorem of course does not guarantee the
existence of a solution on a nontrivial interval  J.  Existence
would follow, for example, if  S  were open.

To treat Example 1, define  $h_1(t, \xi) = \xi$  and  $h_2(t, \xi)$
$= t + \xi$.  For Example 2, let  $h(t, \xi) = t + \xi^{5/3}$  and  $g(\xi) = \xi^{1/3}$.

Proof of the Theorem. Suppose there were two different solutions, $x$ and $y$, on some interval $J = [t_0, b)$ where $b > t_0$. (The case $J = (b, t_0]$ is handled similarly.) Let

$$a \equiv \inf \{t \in (t_0, b) : x(t) \neq y(t)\}.$$

Then $x(a) = y(a)$.

For the point $(a, x(a)) \in S$ let $U, K, m, h_j,$ and $g_j$ be as described in the hypotheses of the theorem. Without loss of generality, assume that for each $j$ the expression in (4) is positive at $(a, x(a))$. Then, reducing $U$ if necessary, the continuity of the derivatives of $h_j$ assures that there exist positive constants $p$ and $M$ such that for $j = 1, \ldots, m$

(5) $\qquad \dfrac{\partial h_j(t,\xi)}{\partial t} + \displaystyle\sum_{i=1}^{n} \dfrac{\partial h_j(t,\xi)}{\partial \xi_i} f_i(t,\xi) \geq p \quad$ on $\quad U \cap S$

and

(6) $\qquad |h_j(t,\xi) - h_j(t,\eta)| \leq M\|\xi - \eta\| \quad$ on $\quad U.$

Choose a bounded interval $[\alpha_j, \beta_j]$ which contains $h_j(U \cap S)$, reducing $U$ if necessary. Then $g_j$ is the difference of two continuous non-decreasing functions on $[\alpha_j, \beta_j]$, and each of the latter can be extended to a continuous non-decreasing function on $R$ by defining it to be constant on $(-\infty, \alpha_j]$ and constant on $[\beta_j, \infty)$. Without loss of generality, we shall assume that each $g_j$ is itself non-decreasing on $R$ and that

(7) $\qquad g_j(h_j(a, x(a))) = 0.$

Define

$$z(t) = \int_a^t \|x'(s) - y'(s)\| \, ds \quad \text{for} \quad a \leq t < b.$$

Then $z(a) = 0$, $z'(a) = 0$, $z$ and $z'$ are continuous, $z'(t) \geq 0$, and $\|x(t) - y(t)\| \leq z(t)$ on $[a, b)$.

Choose $c \in (a, b]$ sufficiently small so that $(s, x(s))$ and $(s, y(s))$ remain in $U$ for $a \leq s < c$. Then, from (6),

$$h_j(s, x(s)) - Mz(s) \leq h_j(s, y(s)) \leq h_j(s, x(s)) + Mz(s)$$

and, from (5),

$$\frac{d}{ds} h_j(s, x(s)) \geq p$$

for $a \leq s < c$ and $j = 1, \ldots, m$.

Thus for $a \leq t < c$, using (3) and the monotonicity of each $g_j$,

$$z(t) \leq K \int_a^t \left\{ ||x(s) - y(s)|| + \sum_{j=1}^m |g_j(h_j(s,x(s))) - g_j(h_j(s,y(s)))| \right\} ds$$

$$\leq K(t - a)z(t) + \frac{K}{p} \sum_{j=1}^m \int_a^t [g_j(h_j(s,x(s)) + Mz(s))$$

$$- g_j(h_j(s,x(s)) - Mz(s))] \frac{d}{ds} h_j(s,x(s)) \, ds$$

$$= K(t - a)z(t) + \frac{K}{p} \sum_{j=1}^m \int_{h_j(t,x(t))-Mz(t)}^{h_j(t,x(t))+Mz(t)} g_j(u) \, du$$

$$- \frac{K}{p} \sum_{j=1}^m \int_a^t [g_j(h_j(s,x(s))+Mz(s)) + g_j(h_j(s,x(s))-Mz(s))]Mz'(s) \, ds.$$

Choose $\delta_1 > 0$ such that for each $j$

$$|g_j(u)| < \frac{p}{6mKM} \quad \text{when} \quad |u - h_j(a,x(a))| < \delta_1.$$

Then choose $\delta \in (0, 1/6K)$ such that $a + \delta \leq c$ and, for each $j$,

$$|h_j(t,x(t)) - h_j(a,x(a))| + Mz(t) < \delta_1 \quad \text{when} \quad a \leq t < a + \delta.$$

Now for $a < t < a + \delta$ one finds $z(t) \leq 5z(t)/6$. This contradiction completes the proof.


The motivation for this paper was the following problem from classical electrodynamics.

Example 3. Consider  n  electrically charged point particles moving along the x-axis at distinct positions, $x_1(t)$, $x_2(t)$, ..., $x_n(t)$. Assume that the motion of particle  j  depends only on the electromagnetic fields produced by the other  n-1  particles, with these fields traveling to particle  j  at the speed of light, c.

The required fields are calculated in terms of the trajectories of the other particles from the retarded Liénard-Wiechert potentials; and they are substituted into the Lorentz force law for particle  j.  Introducing $v_i = x_i'/c$ for the velocity of particle  i  as a multiple of  c, one obtains a system of delay differential equations with state-dependent delays:

$$(8) \qquad \frac{v_j'}{(1 - v_j^2)^{3/2}} = \sum_{i \neq j} \frac{K_{ij}}{r_{ij}^2} \frac{\sigma_{ij} + v_i(t - r_{ij})}{\sigma_{ij} - v_i(t - r_{ij})} ,$$

where each $K_{ij}$ is a constant, $\sigma_{ij} \equiv \text{sgn}\,[x_j(0) - x_i(0)]$, and where $r_{ij} > 0$ satisfies

$$(9) \qquad r_{ij}' = \frac{v_j - v_i(t - r_{ij})}{\sigma_{ij} - v_i(t - r_{ij})} \qquad \text{for } i \neq j.$$

In these equations $v_i$ and $r_{ij}$ without an argument stand for $v_i(t)$ and $r_{ij}(t)$.

In order to solve the system of $n^2$ equations represented by (8) and (9) when $t \geq 0$, one should know not only

$$v_j(0) \text{ and } r_{ij}(0) \text{ for all } j \text{ and all } i \neq j,$$

but also the values of $v_i(t)$ for $t \leq 0$, $i = 1, \ldots, n$.

Now consideration of the problem in three-dimensional motion has led to the conclusion that accelerations should not

be assumed continuous, but only integrable [2]. Thus it seems

reasonable even in the case of one-dimensional motion to assume

that the given past history of $v_i$, say

(11) $\quad v_i(t) = g_i(t) \quad$ for $t \leq 0 \quad (i = 1, \ldots, n)$

is merely absolutely continuous--not, in general, locally

Lipschitzian.

Substituting (11) into the right hand sides of Eqs. (8)

and (9) one gets a system of ordinary differential equations

which satisfies the uniqueness criterion of the present paper.

Thus a unique solution exists at least as long as each

$t - r_{ij}(t) \leq 0 \quad$ and each $\quad |v_j(t)| < 1$. (Further extension

of the solution would use a "method-of-steps" argument which

is not relevant to this paper.)

The above uniqueness problem was solved earlier for the

case of two particles in one-dimensional motion [1]. But the

method used did not seem to extend to the n-body problem.

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